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Journal of Approximation Theory 128 (2004) 175–186

JOURNAL OF Approximation Theory

http://www.elsevier.com/locate/jat

Inequalities of Rafalson type for algebraic polynomials

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Received 21 January 2003; accepted in revised form 27 April 2004

Communicated by Leonid Golinskii

Abstract

For a positive Borel measure $d\mu$, we prove that the constant

$$\gamma_n(dv; d\mu) := \sup_{\pi \in \mathscr{P}_n \setminus \{0\}} \frac{\int_{-\infty}^{\infty} \pi^2(x) dv(x)}{\int_{-\infty}^{\infty} \pi^2(x) d\mu(x)},$$

can be represented by the zeros of orthogonal polynomials corresponding to $d\mu$ in case (i) $dv(x) = (A + Bx)d\mu(x)$, where $A + Bx$ is nonnegative on the support of $d\mu$ and (ii) $dv(x) = (A + Bx^2)d\mu(x)$, where $d\mu$ is symmetric and $A + Bx^2$ is nonnegative on the support of $d\mu$. The extremal polynomials attaining the constant are obtained and some concrete examples are given including Markov-type inequality when $d\mu$ is a measure for Jacobi polynomials.

MSC: 41A17; 41A44

Keywords: Inequalities of Rafalson type; Orthogonal polynomials

1. Introduction

Let $d\mu$ be a positive Borel measure on $\mathbb{R}$ with infinite support whose moments are all finite. Then there exists an orthonormal polynomial system $\{P_n(d\mu; x)\}_{n=0}^{\infty}$ with

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doi:10.1016/j.jat.2004.04.009

respect to $d\mu$ such that

$$\int_{-\infty}^{\infty} P_m(d\mu;x)P_n(d\mu;x)d\mu(x) = \delta_{mn}, \quad m,n = 0,1,2,\ldots,$$

where δ_{mn} is the Kronecker delta. One of the most important properties for $\{P_n(d\mu;x)\}_{n=0}^{\infty}$ is the three term recurrence relation

$$xP_n(d\mu;x) = a_{n+1}P_{n+1}(d\mu;x) + b_nP_n(d\mu;x) + a_nP_{n-1}(d\mu;x), \quad n = 0,1,2,\ldots,$$

where $P_{-1}(x) \equiv 0$, $P_0(d\mu;x) = (\int_{-\infty}^{\infty} d\mu(x))^{\frac{1}{2}}$, and

$$a_n = a_n(d\mu) = \int_{-\infty}^{\infty} xP_n(d\mu;x)P_{n-1}(d\mu;x)d\mu(x), \quad n \geqslant 1,$$

$$b_n = b_n(d\mu) = \int_{-\infty}^{\infty} xP_n^2(d\mu;x)d\mu(x), \quad n \geqslant 0.$$

It is interesting to find the best possible constant $\gamma_n = \gamma_n(dv;d\mu)$ such that

$$||\pi||_{dv} \leqslant \gamma_n ||\pi||_{d\mu}, \quad \pi \in \mathscr{P}_n, \tag{1.1}$$

where $\mathscr{P}_n$ is the space of all real polynomials of degree at most n, dv is another positive Borel measure on $\mathbb{R}$, and

$$||\pi||_{d\mu} := \left\{ \int_{-\infty}^{\infty} \pi^2(x)d\mu(x) \right\}^{\frac{1}{2}}.$$

The constant γ_n can be redefined by

$$\gamma_n(dv;d\mu) = \sup_{\pi \in \mathscr{P}_n} \{||\pi||_{dv} : ||\pi||_{d\mu} = 1\}.$$

For $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx$ and $dv(x) = (1-x)^{\gamma}(1+x)^{\delta}\,dx$ on $[-1,1]$, γ_n was estimated in [1,4] and for $\alpha = \beta = \frac{1}{2}, \gamma = \delta = \frac{3}{2}$ or $\alpha = \beta = \frac{3}{2}, \gamma = \delta = \frac{1}{2}$, the exact value of γ_n was obtained by Rafalson [5].

In this paper, we will prove that the constant γ_n can be expressed by the zeros of orthonormal polynomials with respect to $d\mu$ in cases (i) $dv(x) = (A+Bx)d\mu(x)$, where $A+Bx$ is nonnegative on the support of $d\mu$ and (ii) $dv(x) = (A+Bx^2)d\mu(x)$, where $d\mu$ is symmetric and $A+Bx^2$ is nonnegative on the support of $d\mu$. The extremal polynomial attaining γ_n is obtained and some concrete examples are given including Markov-type inequality when $d\mu$ is a measure for Jacobi polynomials.

2. Case $dv(x) = (A+Bx)d\mu(x)$

The zeros of orthogonal polynomial $P_n(d\mu;x)$ are denoted by $x_{1n}(d\mu) > x_{2n}(d\mu) > \cdots > x_{nn}(d\mu)$. Then by the Gauss quadrature formula, we have

$$x_{1,n+1}(d\mu) = \max_{\pi \in \mathscr{P}_n \backslash \{0\}} \frac{\int_{-\infty}^{\infty} x\pi^2(x)d\mu(x)}{\int_{-\infty}^{\infty} \pi^2(x)d\mu(x)} \tag{2.1}$$

and

$$x_{n+1,n+1}(d\mu) = \min_{\pi\in\mathscr{P}_n\backslash\{0\}} \frac{\int_{-\infty}^{\infty} x\pi^2(x)d\mu(x)}{\int_{-\infty}^{\infty} \pi^2(x)d\mu(x)}. \tag{2.2}$$

The maximum and the minimum in (2.1) and (2.2) are attained if and only if $\pi(x) = \frac{cP_{n+1}(d\mu;x)}{x-x_{1,n+1}(d\mu)}$ and $\pi(x) = \frac{cP_{n+1}(d\mu;x)}{x-x_{n+1,n+1}(d\mu)}$, respectively, where c is a nonzero constant. Using these formula, we can easily prove:

Theorem 2.1. *Let* $dv(x) = g(x)d\mu(x)$, *where* $g(x) = A + Bx$ *is nonnegative on the support of* $d\mu$. *Then*

$$\gamma_n(dv; d\mu) = \left\{\max_{k=1,2,\dots,n+1} g(x_{k,n+1})\right\}^{\frac{1}{2}} = \begin{cases} \sqrt{g(x_{1,n+1})} & \text{if } B\geqslant 0, \\ \sqrt{g(x_{n+1,n+1})} & \text{if } B<0 \end{cases} \tag{2.3}$$

and

$$\gamma_n(d\mu; dv) = \left\{\min_{k=1,2,\dots,n+1} g(x_{k,n+1})\right\}^{-\frac{1}{2}} = \begin{cases} g(x_{n+1,n+1})^{-\frac{1}{2}} & \text{if } B\geqslant 0, \\ g(x_{1,n+1})^{-\frac{1}{2}} & \text{if } B<0, \end{cases} \tag{2.4}$$

where $x_{k,n+1} = x_{k,n+1}(d\mu)$. *The constants* $\gamma_n(dv; d\mu)$ *in* (2.3) *and* $\gamma_n(d\mu; dv)$ *in* (2.4) *are attained if and only if* $\pi(x) = \frac{cP_{n+1}(d\mu;x)}{x-x_{k,n+1}(d\mu)}$, *where c is a nonzero constant and*

$$k = \begin{cases} 1 & \text{if } B\geqslant 0 \\ n+1 & \text{if } B<0 \end{cases} \quad \text{for } \gamma_n(dv; d\mu), \qquad k = \begin{cases} n+1 & \text{if } B\geqslant 0 \\ 1 & \text{if } B<0 \end{cases} \quad \text{for } \gamma_n(d\mu; dv).$$

Proof. By the Gauss quadrature formula, we have for any $\pi\in\mathscr{P}_n$,

$$\begin{aligned} \int_{-\infty}^{\infty} \pi^2(x)dv(x) &= \int_{-\infty}^{\infty} (A + Bx)\pi^2(x)d\mu(x) \\ &= \sum_{k=1}^{n+1} \lambda_{k,n+1}(A + Bx_{k,n+1})\pi^2(x_{k,n+1}) \\ &\leqslant \max_{k=1,2,\dots,n+1} (A + Bx_{k,n+1}) \sum_{k=1}^{n+1} \lambda_{k,n+1}\pi^2(x_{k,n+1}) \\ &= \max_{k=1,2,\dots,n+1} g(x_{k,n+1}) \int_{-\infty}^{\infty} \pi^2(x)d\mu(x), \end{aligned} \tag{2.5}$$

where $\lambda_{k,n+1} := \lambda_{k,n+1}(d\mu)$ are the Christoffel numbers for the measure $d\mu$. Now assume $B\geqslant 0$. Then $\max_{k=1,2,\dots,n+1} g(x_{k,n+1}) = g(x_{1,n+1})$ and we have the equality in (2.5) for $\pi(x) = \frac{P_{n+1}(x)}{x-x_{1,n+1}}$. Conversely if the equality holds in (1.1) for $\pi(x)$, then the equality holds also in (2.5) so that $\pi(x_{k,n+1}) = 0,\ 2\leqslant k\leqslant n+1$. Hence $\pi(x) = \frac{cP_{n+1}(x)}{x-x_{1,n+1}},\ c\neq 0$. This proves (2.3) when $B\geqslant 0$. In case $B<0$, the proof is similar. Finally

Eq. (2.4) can be proved by a similar process using (2.2) instead of (2.1) and

$$\gamma_n^2(d\mu; d\nu) = \max_{\pi\in\mathscr{P}_n\setminus\{0\}} \frac{\int_{-\infty}^{\infty} \pi^2(x)d\mu(x)}{\int_{-\infty}^{\infty} \pi^2(x)d\nu(x)} = \left\{ \min_{\pi\in\mathscr{P}_n\setminus\{0\}} \frac{\int_{-\infty}^{\infty} \pi^2(x)d\nu(x)}{\int_{-\infty}^{\infty} \pi^2(x)d\mu(x)} \right\}^{-1}. \qquad \square \tag{2.6}$$

Corollary 2.2. *Let* $d\nu(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, $d\mu(x) = (1-x)^{\gamma}(1+x)^{\delta}\,dx$ *on* $[-1,1]$, *and* $\varphi_{\gamma,\delta}^{\alpha,\beta}(n) = \gamma_n(d\nu; d\mu)$, *where* $\alpha,\beta,\gamma,\delta > -1$. *Then*

$$\varphi_{1/2,1/2}^{3/2,1/2}(n) = \varphi_{1/2,1/2}^{1/2,3/2}(n) = \sqrt{2}\cos\frac{\pi}{2(n+2)};$$

$$\varphi_{3/2,1/2}^{1/2,1/2}(n) = \varphi_{1/2,3/2}^{1/2,1/2}(n) = \left(\sqrt{2}\sin\frac{\pi}{2(n+1)}\right)^{-1};$$

$$\varphi_{-1/2,-1/2}^{1/2,-1/2}(n) = \varphi_{-1/2,-1/2}^{-1/2,1/2}(n) = \sqrt{2}\cos\frac{\pi}{4(n+1)};$$

$$\varphi_{1/2,-1/2}^{-1/2,-1/2}(n) = \varphi_{-1/2,1/2}^{-1/2,-1/2}(n) = \left(\sqrt{2}\sin\frac{\pi}{4(n+1)}\right)^{-1}.$$

Proof. Let $g(x) = 1-x$. Since the smallest zero of Chebychev polynomial $U_{n+1}(x)$ of the second kind is $-\cos\frac{\pi}{n+2}$,

$$\varphi_{1/2,1/2}^{3/2,1/2}(n) = \sqrt{1+\cos\frac{\pi}{n+2}} = \sqrt{2}\cos\frac{\pi}{2(n+2)}. \tag{2.7}$$

All others can be proved similarly by Theorem 2.1. $\square$

Example 2.1. Let $d\mu(x) = x^{\alpha}e^{-x}\,dx$ and $d\nu(x) = x\,d\mu(x)$ on $[0,\infty)$, where $\alpha > -1$. Using the asymptotic behavior of the greatest zero $x_{1,n+1}$ of the Laguerre polynomial $L_{n+1}^{(\alpha)}(x)$ [6], we can use

$$\lim_{n\to\infty} \frac{\gamma_n(d\nu; d\mu)}{2\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{x_{1,n+1}}}{2\sqrt{n}} = 1.$$

Let $d\nu(x) = g(x)d\mu(x)$, where $g\in\mathscr{P}_\ell$ is nonnegative on $[0,\infty)$. Then by the same process as in the proof of Theorem 2.1, we have for any $\pi\in\mathscr{P}_n$,

$$\int_0^{\infty} \pi^2(x)d\nu(x) \leqslant \max_{k=1,2,\ldots,n+m} g(x_{k,n+m}) \int_0^{\infty} \pi^2(x)d\mu(x), \quad m = \left[\frac{\ell+1}{2}\right]$$

and

$$\int_0^\infty \pi^2(x)dv(x) \geqslant \min_{k=1,2,\dots,n+m} g(x_{k,n+m}) \int_0^\infty \pi^2(x)d\mu(x), \quad m = \left[\frac{\ell+1}{2}\right].$$

Hence, we obtain an estimation for $\gamma_n(dv; d\mu)$:

$$\min_{k=1,2,\dots,n+m} g(x_{k,n+m}) \leqslant \gamma_n^2(dv; d\mu) \leqslant \max_{k=1,2,\dots,n+m} g(x_{k,n+m}). \tag{2.8}$$

But, estimate (2.8) is not sharp in general if $l \geqslant 2$.

3. Case $d\mathbf{v}(x) = (A + Bx^2)d\boldsymbol{\mu}(x)$

In this section, $d\mu$ is assumed to be symmetric and so the corresponding orthonormal polynomials satisfy

$$xP_n(d\mu; x) = a_{n+1}P_{n+1}(d\mu; x) + a_nP_{n-1}(d\mu; x), \quad n \geqslant 0.$$

Lemma 3.1. *Let $d\mu$ be symmetric. Then we have*

$$x_{1,n+2}(d\mu) = \max_{\pi \in \mathscr{P}_n \setminus \{0\}} \frac{\int_{-\infty}^{\infty} x^2\pi^2(x)d\mu(x)}{\int_{-\infty}^{\infty} \pi^2(x)d\mu(x)}$$

and equality holds if and only if $\pi(x) = \frac{cP_{n+2}(d\mu;x)}{x^2 - x_{1,n+2}^2}$, *where c is a nonzero constant.*

Proof. See Theorem 2 in [2]. □

Lemma 3.2. *For any $(n+1) \times (n+1)$ matrix W $(n \geqslant 1)$,*

$$W := \begin{pmatrix} \alpha_0 & 0 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 & 0 & \beta_2 & 0 & 0 & \cdots & 0 \\ \beta_1 & 0 & \alpha_2 & 0 & \beta_3 & 0 & \cdots & 0 \\ 0 & \beta_2 & 0 & \alpha_3 & 0 & \beta_4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \beta_{n-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \alpha_{n-1} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_{n-1} & 0 & \alpha_n \end{pmatrix} \tag{3.1}$$

we have $|W| = |U||V|$, *where* $|W|$ *is the determinant of the matrix* W,

$$U := \begin{pmatrix} \alpha_0 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_3 & 0 & 0 & \cdots & 0 \\ 0 & \beta_3 & \alpha_4 & \beta_5 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \alpha_{2m-2} & \beta_{2m-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_{2m-1} & \alpha_{2m} \end{pmatrix}, \quad m := \left[\frac{n}{2}\right]$$

and

$$V := \begin{pmatrix} \alpha_1 & \beta_2 & 0 & 0 & 0 & \cdots & 0 \\ \beta_2 & \alpha_3 & \beta_4 & 0 & 0 & \cdots & 0 \\ 0 & \beta_4 & \alpha_5 & \beta_6 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \alpha_{2\ell-1} & \beta_{2\ell} \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_{2\ell} & \alpha_{2\ell+1} \end{pmatrix}, \quad \ell := \left[\frac{n-1}{2}\right].$$

Proof. We only prove the case $n = 2m$ even since the other case can be proved by same way. Let

$$W = [C_0, C_1, \dots, C_n],$$

where C_i is the ith column of W. By moving every even column of W to the right, we obtain

$$W_1 = [C_0, C_2, \dots, C_n, C_1, C_3, \dots, C_{n-1}].$$

Write the transpose W_1^T of W_1 as

$$W_1^T = [C_0^1, C_1^1, \dots, C_n^1],$$

where C_i^1 is the ith column of W_1^T. By moving every even column of W_1^T to the right, we obtain

$$\begin{aligned} W_2 &= [C_0^1, C_2^1, \dots, C_n^1, C_1^1, C_3^1, \dots, C_{n-1}^1] \\ &= \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}. \end{aligned}$$

Then $|W| = |W_2| = |U||V|$. $\square$

Theorem 3.3. *Let* $dv(x) = (A + Bx^2)d\mu(x)$, *where* $A + Bx^2$ *is nonnegative on the support of* $d\mu$. *If* $d\mu$ *is symmetric, then*

$$\gamma_n(dv; d\mu) = \begin{cases} \sqrt{A + Bx^2_{1,n+2}} & \text{if } B \geqslant 0, \\ \sqrt{A + Bx^2_{s+1,n+2}} & \text{if } B < 0 \text{ and } n = 2s, \\ \sqrt{A + Bx^2_{s+1,n+1}} & \text{if } B < 0 \text{ and } n = 2s + 1 \end{cases} \tag{3.2}$$

and

$$\gamma_n(d\mu; dv) = \begin{cases} (A + Bx^2_{1,n+2})^{-\frac{1}{2}} & \text{if } B \leqslant 0, \\ (A + Bx^2_{s+1,n+2})^{-\frac{1}{2}} & \text{if } B > 0 \text{ and } n = 2s, \\ (A + Bx^2_{s+1,n+1})^{-\frac{1}{2}} & \text{if } B > 0 \text{ and } n = 2s + 1. \end{cases} \tag{3.3}$$

Proof. We will prove only (3.2). Then (3.3) can be proved by a similar process with (2.6). When $B = 0$, it is trivial and so we may assume $B \neq 0$. Let $\pi(x) = \sum_{k=0}^{n} c_k P_k(d\mu; x)$. Then by the three term recurrence relation,

$$\begin{aligned}(A + Bx^2)\pi(x) &= \sum_{k=0}^{n} (A + Bx^2)c_k P_k(x) \\ &= \sum_{k=0}^{n} [A + B(a^2_{k+1} + a^2_k)]c_k P_k(x) \\ &\quad + \sum_{k=2}^{n+2} Ba_k a_{k-1} c_{k-2} P_k(x) + \sum_{k=0}^{n-2} Ba_{k+2} a_{k+1} c_{k+2} P_k(x),\end{aligned}$$

where $a_k = a_k(d\mu)$ and $P_k(x) = P_k(d\mu; x)$. Hence, by the orthonormality of $\{P_n(x)\}_{n=0}^{\infty}$,

$$\int_{-\infty}^{\infty} \pi^2(x)dv(x) = \sum_{k=0}^{n} [A + B(a^2_{k+1} + a^2_k)]c^2_k + 2\sum_{k=0}^{n-2} Ba_{k+2}a_{k+1}c_k c_{k+2}.$$

If we assume that $||\pi||_{d\mu} = 1$, that is, $\sum_{k=0}^{n} c_k^2 = 1$, then

$$\gamma^2_n(dv; d\mu) = \max_{\sum_{k=0}^{n} c_k^2 = 1} \left\{ \sum_{k=0}^{n} [A + B(a^2_{k+1} + a^2_k)]c^2_k + 2\sum_{k=0}^{n-2} Ba_{k+2}a_{k+1}c_k c_{k+2} \right\},$$

which is equal to $\max\{|\lambda| : \lambda \text{ is an eigenvalue of } W\}$, where W is matrix (3.1) with $\alpha_k = A + B(a_k^2 + a_{k+1}^2)$ and $\beta_k = Ba_k a_{k+1}$. By Lemma 3.2, $\gamma_n(dv : d\mu) = \max\{|\lambda| :$

$U_m(\lambda) = 0$ or $V_\ell(\lambda) = 0\}$, where

$$U_m(\lambda) = \begin{vmatrix} \alpha_0 - \lambda & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 - \lambda & \beta_3 & 0 & 0 & \cdots & 0 \\ 0 & \beta_3 & \alpha_4 - \lambda & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \alpha_{2m-2} - \lambda & \beta_{2m-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_{2m-1} & \alpha_{2m} - \lambda \end{vmatrix} \left(m = \left[\frac{n}{2}\right]\right)$$

and

$$V_\ell(\lambda) = \begin{vmatrix} \alpha_1 - \lambda & \beta_2 & 0 & 0 & 0 & \cdots & 0 \\ \beta_2 & \alpha_3 - \lambda & \beta_4 & 0 & 0 & \cdots & 0 \\ 0 & \beta_4 & \alpha_5 - \lambda & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \alpha_{2\ell-1} - \lambda & \beta_{2\ell} \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_{2\ell} & \alpha_{2\ell+1} - \lambda \end{vmatrix}$$
$$\left(\ell = \left[\frac{n-1}{2}\right]\right).$$

Now zeros of $U_m(\lambda)$ and $V_\ell(\lambda)$ are the zeros of orthonormal polynomials $S_{m+1}(x)$ and $T_{\ell+1}(x)$, respectively, satisfying

$$xS_k = Ba_{2k+2}a_{2k+1}S_{k+1} + [A + B(a_{2k+1}^2 + a_{2k}^2)]S_k + Ba_{2k}a_{2k-1}S_{k-1}, \tag{3.4}$$

$$xT_k = Ba_{2k+3}a_{2k+2}T_{k+1} + [A + B(a_{2k+2}^2 + a_{2k+1}^2)]T_k + Ba_{2k+1}a_{2k}T_{k-1}. \tag{3.5}$$

On the other hand, since $d\mu$ is symmetric, if we set

$$Q_k(x^2) = P_{2k}(d\mu; x) \text{ and } xR_k(x^2) = P_{2k+1}(d\mu; x), \quad k \geqslant 0, \tag{3.6}$$

then $\{Q_k(x)\}_{k=0}^{\infty}$ and $\{R_k(x)\}_{k=0}^{\infty}$ are orthonormal polynomials satisfying the three term recurrence relations

$$\begin{aligned} xQ_k(x) = {} & a_{2k+2}a_{2k+1}Q_{k+1}(x) + (a_{2k+1}^2 + a_{2k}^2)Q_k(x) \\ & + a_{2k}a_{2k-1}Q_{k-1}(x), \quad k \geqslant 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} xR_k(x) = {} & a_{2k+3}a_{2k+2}R_{k+1}(x) + (a_{2k+2}^2 + a_{2k+1}^2)R_k(x) \\ & + a_{2k+1}a_{2k}R_{k-1}(x), \quad k \geqslant 0. \end{aligned} \tag{3.8}$$

Then $\{Q_n(\frac{1}{B}(x - A))\}_{n=0}^{\infty}$ and $\{R_n(\frac{1}{B}(x - A))\}_{n=0}^{\infty}$ satisfy the recurrence relations (3.4) and (3.5), respectively. Hence,

$$S_{m+1}(x) = Q_{m+1}\left(\frac{1}{B}(x - A)\right) \text{ and } T_{\ell+1}(x) = R_{\ell+1}\left(\frac{1}{B}(x - A)\right).$$

From relation (3.6), $Q_{m+1}(x^2_{k,2m+2}) = 0$, $k = 1, 2, \ldots, m+1$, and $R_{\ell+1}(x^2_{k,2\ell+3}) = 0$, $k = 1, 2, \ldots, \ell+1$ and so

$$S_{m+1}(A + Bx^2_{k,2m+2}) = 0, \quad k = 1, 2, \ldots, m+1,$$

$$T_{\ell+1}(A + Bx^2_{k,2\ell+3}) = 0, \quad k = 1, 2, \ldots, \ell+1.$$

Hence

$$\gamma_n^2(dv; d\mu) = \max_{\substack{k=1,2,\ldots,m+1; \\ j=1,2,\ldots,\ell+1}} \{A + Bx^2_{k,2m+2}, A + Bx^2_{j,2\ell+3}\}$$
$$= \begin{cases} A + B\max\{x^2_{1,2m+2}, x^2_{1,2\ell+3}\} & \text{if } B>0 \\ A + B\min\{x^2_{m+1,2m+2}, x^2_{\ell+1,2\ell+3}\} & \text{if } B<0. \end{cases}$$

If $B>0$ and $n = 2s$ is even, then $m = s$ and $\ell = s - 1$ so that

$$\gamma_n^2(dv; d\mu) = A + Bx^2_{1,2s+2} = A + Bx^2_{1,n+2}.$$

If $B>0$ and $n = 2s+1$ is odd, then $m = s$ and $\ell = s$ so that

$$\gamma_n^2(dv; d\mu) = A + Bx^2_{1,2s+3} = A + Bx^2_{1,n+2}.$$

If $B<0$ and $n = 2s$ is even, then $m = s$ and $\ell = s - 1$ so that

$$\gamma_n^2(dv; d\mu) = A + Bx^2_{s+1,2s+2} = A + Bx^2_{s+1,n+2}$$

since $0<x_{s+1,n+1}<x_{s,n+1}$. If $B<0$ and $n = 2s+1$ is odd,then $m = \ell = s$ so that

$$\gamma_n^2(dv; d\mu) = A + Bx^2_{s+1,2s+2} = A + Bx^2_{s+1,n+1}.$$

since $0<x_{s+1,n+1}<x_{s+1,n+2}$. Hence, the conclusion follows. □

Note that the constant $\gamma_n(dv; d\mu)$ in (2.2) is attained if and only if

$$\pi(x) = \begin{cases} \dfrac{cP_{n+2}(d\mu; x)}{x^2 - x^2_{1,n+2}} & \text{if } B\geqslant 0, \\ \dfrac{cQ_{s+1}(x)}{x^2 - x^2_{s+1,2s+2}} & \text{if } B<0, \end{cases}$$

where c is a nonzero constant.

Corollary 3.4. *Let* $dv(x) = (1-x)^\alpha(1+x)^\beta\,dx$, $d\mu(x) = (1-x)^\gamma(1+x)^\delta\,dx$ *on* $[-1, 1]$, *and* $\varphi^{\alpha,\beta}_{\gamma,\delta}(n) = \gamma_n(dv; d\mu)$, *where* $\alpha, \beta, \gamma, \delta > -1$. *Then*

$$\varphi^{3/2,3/2}_{1/2,1/2}(n) = \begin{cases} \cos\dfrac{\pi}{2(n+3)} & \text{if } n \text{ is even}, \\ \cos\dfrac{\pi}{2(n+2)} & \text{if } n \text{ is odd}, \end{cases}$$

$$\varphi^{1/2,1/2}_{3/2,3/2}(n) = \left(\sin\frac{\pi}{n+3}\right)^{-1},$$

$$\varphi_{-1/2,-1/2}^{1/2,1/2}(n) = \begin{cases} \cos\dfrac{\pi}{2(n+2)} & \text{if } n \text{ is even,} \\ \cos\dfrac{\pi}{2(n+1)} & \text{if } n \text{ is odd,} \end{cases}$$

$$\varphi_{1/2,1/2}^{-1/2,-1/2}(n) = \left(\sin\frac{\pi}{2(n+2)}\right)^{-1}.$$

Proof. If $\alpha = \beta = \frac{3}{2}$ and $\gamma = \delta = \frac{1}{2}$, then $dv(x) = (1-x^2)d\mu(x)$ and the orthonormal polynomials $\{U_n(x)\}_{n=0}^{\infty}$ with respect to $d\mu$ are the Chebychev polynomials of the second kind, whose zeros are

$$x_{kn}(d\mu) = \cos\frac{k\pi}{n+1}, \quad k = 1, 2, \ldots, n.$$

Hence, by Theorem 3.3, if $n = 2s$, then

$$\varphi_{1/2,1/2}^{3/2,3/2}(n) = \sqrt{1 - \cos^2\frac{(s+1)\pi}{2s+3}} = \cos\frac{\pi}{2(n+3)}$$

and if $n = 2s+1$, then

$$\varphi_{1/2,1/2}^{3/2,3/2}(n) = \sqrt{1 - \cos^2\frac{(s+1)\pi}{2s+3}} = \cos\frac{\pi}{2(n+2)}.$$

All the other cases can be obtained similarly by Theorem 3.3 and the zeros of the Chebychev polynomials of the first and the second kinds. □

Corollary 3.5. *Let $d\mu$ be symmetric. Then we have*

$$\min_{\pi \in \mathscr{P}_n \setminus \{0\}} \frac{\int_{-\infty}^{\infty} x^2\pi^2(x)d\mu(x)}{\int_{-\infty}^{\infty} \pi^2(x)d\mu(x)} = \begin{cases} x_{s+1,n+2}^2 & \text{if } n = 2s, \\ x_{s+1,n+1}^2 & \text{if } n = 2s+1. \end{cases} \tag{3.9}$$

The minimum is attained if and only if $\pi(x) = \frac{cP_{n+2}(d\mu;x)}{x^2 - x_{s+1,n+2}^2(d\mu)}$ *when* $n = 2s$ *and* $\pi(x) = \frac{cP_{n+1}(d\mu;x)}{x^2 - x_{s+1,n+1}^2(d\mu)}$ when $n = 2s+1$, where c is a nonzero constant.

Proof. Take $A = 0$ and $B = 1$ in Theorem 3.3. Then $\min_{\pi \in \mathscr{P}_n \setminus \{0\}} \frac{\int_{-\infty}^{\infty} x^2\pi^2(x)d\mu(x)}{\int_{-\infty}^{\infty} \pi^2(x)d\mu(x)} = \gamma_n^{-2}(d\mu; dv)$ and so (3.9) holds by Theorem 3.3. By the Gauss quadrature formula and (2.6), we can show that the minimum is attained only when

$$\pi(x) = \begin{cases} \dfrac{cP_{n+2}(d\mu;x)}{x^2 - x_{s+1,n+2}^2(d\mu)} & \text{if } n = 2s, \\ \dfrac{cP_{n+1}(d\mu;x)}{x^2 - x_{s+1,n+1}^2(d\mu)} & \text{if } n = 2s+1, \end{cases}$$

where c is a nonzero constant. □

The following sharp inequality was proved in [3] (see also [1] for $\alpha=\beta$). If $\pi\in\mathscr{P}_n$ and $\alpha,\beta>-1$, then

$$||\pi^{(m)}||_{m+\alpha,m+\beta}\leqslant\sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}||\pi||_{\alpha,\beta},\tag{3.10}$$

where

$$||\pi||_{\alpha,\beta}=\left(\int_{-1}^{1}\pi^2(x)(1-x)^\alpha(1+x)^\beta dx\right)^{\frac{1}{2}}.$$

Applying Theorem 2.1 iteratively, if $\alpha=\beta+k$, then

$$\begin{aligned}||\pi^{(m)}||_{\alpha+m,\beta+m}&\leqslant\sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}\,||\pi||_{\beta+k,\beta}\\&\leqslant\sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}\prod_{j=0}^{k-1}\varphi_{\beta+j,\beta}^{\beta+j+1,\beta}(n)\ \ ||\pi||_{\beta,\beta}\\&=\sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}\prod_{j=0}^{k-1}(1-x_{n+1,n+1}^{\beta+j,\beta})^{\frac{1}{2}}||\pi||_{\beta,\beta},\end{aligned}\tag{3.11}$$

where $\{x_{k,n}^{\alpha,\beta}\}_{k=1}^{n}$ denotes the zeros of Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. Similarly, if $\alpha=\beta-k$, then

$$||\pi^{(m)}||_{\alpha+m,\beta+m}\leqslant\sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}\prod_{j=0}^{k-1}\ (1+x_{1,n+1}^{\alpha,\alpha+j})^{\frac{1}{2}}||\pi||_{\alpha,\alpha}.$$

Combining Theorem 3.3 and applying Theorem 2.1 again, we obtain a Markov type inequality. More precisely, if $\alpha=\beta+k$, then

$$\begin{aligned}||\pi^{(m)}||_{\alpha+m,\beta+m}&\leqslant D_{n,m}^{\alpha,\beta}\prod_{j=0}^{k-1}\ \sqrt{1-x_{n+1,n+1}^{\beta+j,\beta}}\,||\pi||_{\beta,\beta}\\&\leqslant D_{n,m}^{\alpha,\beta}\prod_{j=0}^{k-1}\ \sqrt{1-x_{n+1,n+1}^{\beta+j,\beta}}\prod_{j=0}^{m-1}(1-(x_{1,n+2}^{\beta+j,\beta+j})^2)^{-\frac{1}{2}}\,||\pi||_{\beta+k+m,\beta+m}\\&=D_{n,m}^{\alpha,\beta}\prod_{j=0}^{k-1}\ \sqrt{\frac{1-x_{n+1,n+1}^{\beta+j,\beta}}{1+x_{1,n+1}^{\beta+m+j,\beta+m}}}\prod_{j=0}^{m-1}(1-(x_{1,n+2}^{\beta+j,\beta+j})^2)^{-\frac{1}{2}}\,||\pi||_{\alpha+m,\beta+m}\end{aligned}$$

and if $\alpha = \beta - k$, then

$$||\pi^{(m)}||_{\alpha+m,\beta+m} \leqslant D^{\alpha,\beta}_{n,m} \prod_{j=0}^{k-1} \sqrt{\frac{1 + x^{\alpha,\alpha+j}_{1,n+1}}{1 + x^{\alpha+m,\alpha+m+j}_{n+1,n+1}}} \times \prod_{j=0}^{m-1} (1 - (x^{\alpha+j,\alpha+j}_{1,n+2})^2)^{-\frac{1}{2}} \, ||\pi||_{\alpha+m,\beta+m}, \tag{3.12}$$

where

$$D^{\alpha,\beta}_{n,m} = \sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}.$$

In particular, if $k = 0$, then $\alpha = \beta$ and

$$||\pi^{(m)}||_{\alpha+m,\beta+m} \leqslant \sqrt{\frac{n!\Gamma(n+m+2\alpha+1)}{(n-m)!\Gamma(n+2\alpha+1)}} \prod_{j=0}^{m-1} (1 - (x^{\alpha+j,\alpha+j}_{1,n+2})^2)^{-\frac{1}{2}} ||\pi||_{m+\alpha,m+\alpha},$$

which is a Markov-type inequality for ultraspherical polynomials. As a special case, we obtain ($\alpha = \beta = -\frac{1}{2}$ and $m = 1$)

$$||\pi'||_{\frac{1}{2},\frac{1}{2}} \leqslant \frac{n}{\sin \frac{\pi}{2(n+2)}} ||\pi||_{\frac{1}{2},\frac{1}{2}},$$

which was also found in [5]. In this way, we can obtain various kinds of inequalities using (3.10), (3.11), and (3.12).

Acknowledgments

K.H. Kwon was partially supported by KRF-2002-070-C00004 and D.W. Lee was supported by the Post-doctoral Fellowship Program of KOSEF. In the case studied in Section 3 it can be seen easily, as the first step of the proof, that the extremal polynomial is either even or odd. Hence the main result in Section 3 can be derived from that of Section 2. Authors are grateful to the referee for this observation.

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